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ON THE NONSINGULARITY OF LINEAR COMBINATIONS OF GENERALIZED PROJECTORS

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Abstract
This paper investigates the nonsingularity of linear combinations of generalized projectors. We establish necessary and sufficient conditions under which a linear combination of such projectors is invertible. In addition, explicit expressions for the inverse are derived for cases where invertibility holds. The presented results generalize known facts for classical projectors and provide a systematic framework for analyzing linear combinations in terms of their algebraic and spectral properties. Illustrative examples demonstrate the applicability of the obtained formulas and conditions.
Keywords: nonsingularity, invertibility conditions, linear combination, generalized projector.

INTRODUCTION

Let $\mathbb{C}^{n\times m}$ denote the set of all $n\times m$ complex matrices. For a matrix $A\in\mathbb{C}^{n\times m}$, we use the symbols A^*, $R(A), N(A)$ and $r(A)$ to denote the conjugate transpose matrix, range, the null space and rank of A, respectively. By $\mathbb{C}_r^{n\times n}$ we will denote for the set of all matrices from $\mathbb{C}^{n\times n}$ with a rank r.
The identity matrix of order n will be denoted by I_n. The symbol $\oplus$ will be used for direct sum.
A square matrix A is said to be normal if it commutes with its conjugate transpose, that is, if $AA^*=A^*A$. The notation C_n^N denotes the set of all normal matrices of order n.
An important class of square matrices is formed by projectors and, in particular, by orthogonal projectors. Owing to their structural properties, these matrices are widely used in matrix decompositions, in the characterization of subspaces, and in the formulation of generalized inverses. The matrix $P\in\mathbb{C}^{n\times n}$ satisfying $P^2=P$ is called the projector (the idempotent matrix), until the matrix $P\in\mathbb{C}^{n\times n}$ satisfying $P^2=P=P^*$ is called the orthogonal projector. P_S denotes the orthogonal projector onto subspace S.
A natural generalization of these matrices is provided by generalized projectors, introduced in 1997 by Groβ and Trenkler [1]. The generalized projector is a square matrix such that $A^2=A^*$.
We use the notation C_n^{GP} for the subsets of $\mathbb{C}^{n\times n}$ consisting of generalized projectors.
In the following years, various properties of generalized inverses were intensively studied, as reported in $[2-7]$. Among these, the invertibility of generalized projectors has received particular attention, which motivates the present study. Necessary and sufficient conditions for the invertibility of linear combinations of generalized projectors are presented, along with new explicit forms of the corresponding inverses.

EXPOSITION

The following form of the generalized inverse enables us to derive results

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concerning the invertibility of a linear combination of generalized projectors..Any generalized projector $A \in \mathbb{C}_r^{n\times n}$ can be represented by

$$A = U\begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} U^*, \qquad (1)$$

where $U \in \mathbb{C}^{n\times n}$ is unitary and $K \in \mathbb{C}^{r\times r}$ is such that $K^3 = I_r$ and $K^* = K^{-1}$. Based on this form, a representation of the inverse of a linear combination of generalized projectors can be derived, in which the sum of the involved projectors is itself a generalized projector.

Theorem 1. [6] Let $A \in \mathbb{C}_r^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ be generalized projectors, and let $k, l \in \mathbb{N}$, $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. If $A + B \in C_n^{GP}$, then the following conditions are equivalent:

(i) $R(A) \oplus R(B) = \mathbb{C}^{n\times 1}$,

(ii) $N(A) \oplus N(B) = \mathbb{C}^{n\times 1}$,

(iii) $R(A) \cap R(B) = \{0\}$, $N(A) \cap N(B) = \{0\}$,

(iv) $c_1A^k + c_2B^l$ is nonsingular.

Furthermore, if one of the conditions $((i) - (iv)$ holds, then

$$(c_1A^k + c_2B^l)^{-1} = U\begin{bmatrix} c_1^{-1}K^{-k} & 0 \\ 0 & c_2^{-1}G^{-l} \end{bmatrix} U^*, \qquad (1)$$

where $U \in \mathbb{C}^{n\times n}$ is unitary, $K \in \mathbb{C}^{r\times r}$ is such that $K^3 = I_r$, $K^* = K^{-1}$,

$$K^{-k} = \begin{cases} I_r, k \equiv_3 0 \\ K^*, k \equiv_3 1, \\ K, k \equiv_3 2 \end{cases}$$

and $G \in \mathbb{C}^{(n-r)\times(n-r)}$ is an invertible generalized projector such that

$$G^{-l} = \begin{cases} I_r, l \equiv_3 0 \\ G^*, l \equiv_3 1. \\ G, l \equiv_3 2 \end{cases}$$

Observe that formula (1) admits a more elegant representation. Indeed, since for two generalized projectors A, B, their sum $A + B$ is a generalized projector if and only if

$$A = U\begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} U^*, \quad (2)$$

$$B = U\begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix} U^*, \quad (3)$$

it follows that

$$U\begin{bmatrix} c_1^{-1}K^{-k} & 0 \\ 0 & c_2^{-1}G^{-l} \end{bmatrix} U^* = \frac{1}{c_1}A^{2k} + \frac{1}{c_2}B^{2l}.$$

Thus,

$$(c_1A^k + c_2B^l)^{-1} = \frac{1}{c_1}A^{2k} + \frac{1}{c_2}B^{2l}.$$

As a consequence of this theorem, the necessary and sufficient conditions for the invertibility of the sum and the difference of generalized projectors are derived, in the case when their sum is also a generalized projector, together with the corresponding representations of their inverses.

Corollary 2. [6] Let $A \in \mathbb{C}_r^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ be generalized projectors, and let $k, l \in \mathbb{N}$, $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. If $A + B \in C_n^{GP}$, then the following conditions are equivalent:

(i) $R(A) \oplus R(B) = \mathbb{C}^{n\times 1}$,

(ii) $N(A) \oplus N(B) = \mathbb{C}^{n\times 1}$,

(iii) $R(A) \cap R(B) = \{0\}$, $N(A) \cap N(B) = \{0\}$,

(iv) $A - B$ is nonsingular,

(iv) $A + B$ is nonsingular.

Furthermore, if one of the conditions $((i) - (iv)$ holds, then

$$(A - B)^{-1} = U\begin{bmatrix} K^{-1} & 0 \\ 0 & -G^{-1} \end{bmatrix} U^* = U\begin{bmatrix} K^* & 0 \\ 0 & -G^* \end{bmatrix} U^*, \qquad (2)$$

and

$$(A + B)^{-1} = U\begin{bmatrix} K^{-1} & 0 \\ 0 & G^{-1} \end{bmatrix} U^* = U\begin{bmatrix} K^* & 0 \\ 0 & G^* \end{bmatrix} U^*, \qquad (3)$$

where $U \in \mathbb{C}^{n\times n}$ is unitary, $K \in \mathbb{C}^{r\times r}$ is such that $K^3 = I_r$, $K^* = K^{-1}$, and and $G \in \mathbb{C}^{(n-r)\times(n-r)}$ is an invertible generalized projector.

In this context, the inverse of both the sum and the difference of generalized projectors can be expressed in a more elegant form. Employing forms (2) and (3), it can be concluded that

$$U\begin{bmatrix} K^{-1} & 0 \\ 0 & -G^{-1} \end{bmatrix} U^* = U\begin{bmatrix} K^* & 0 \\ 0 & -G^* \end{bmatrix} U^* = A^2 - B^2$$

and

$$U\begin{bmatrix}K^{-1} & 0\\ 0 & G^{-1}\end{bmatrix}U^* = U\begin{bmatrix}K^* & 0\\ 0 & G^*\end{bmatrix}U^* \\ = A^2 + B^2.$$

Therefore,

$$(A-B)^{-1} = A^2 - B^2$$

and

$$(A+B)^{-1} = A^2 + B^2.$$

Example 1. Let $A, B \in \mathbb{C}^{2\times 2}$ be defined by

$$A = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},$$

$$B = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}.$$

By direct calculation, we obtain $A^2 = A^* = A$, and $B^2 = B^* = B$, hence, $A, B \in C_n^{GP}$. Also, $AB = 0 = BA$, therefore $A + B \in C_n^{GP}$. First, the sum

$$A + B = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = I_2,$$

which is clearly invertible. By Corollary 2,

$$(A+B)^{-1} = A^2 + B^2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = I_2.$$

Next, the difference

$$A - B = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

Is also invertible, and its inverse is given by

$$(A-B)^{-1} = A^2 - B^2 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

Thus, for generalized projectors A and B with $AB = 0 = BA$, both the sum $A + B$ and the difference $A - B$ are invertible, with inverses given by

$$(A+B)^{-1} = A^2 + B^2,$$
$$(A-B)^{-1} = A^2 - B^2,$$

as stated in Corollary 2.

The following theorem establishes the necessary and sufficient conditions under which two commuting generalized projectors lead to the simultaneous invertibility of the corresponding expressions.

Theorem 3. [6] Let $A, B \in \mathbb{C}^{n\times n}$ be generalized projectors and $AB = BA$. The following conditions are equivalent:
(i) $R(A) \oplus R(B) = \mathbb{C}^{n\times 1}$,
(ii) $N(A) \oplus N(B) = \mathbb{C}^{n\times 1}$,
(iii) $R(A) \cap R(B) = \{0\}$ and $N(A) \cap N(B) = \{0\}$,
(iv) $A - B,\ A^* + AB + B^*$ are nonsingular.

When the product of two generalized projectors is a generalized projector or a normal matrix, it follows that they commute. Thus, the next assertion holds.

Corollary 4. [6] Let $A, B \in C_n^{GP}$ be such that $AB \in C_n^{GP}$ (or $AB \in C_n^{N}$) and one of A and B is idempotent. The following conditions are equivalent:
(i) $R(A) \oplus R(B) = \mathbb{C}^{n\times 1}$,
(ii) $N(A) \oplus N(B) = \mathbb{C}^{n\times 1}$,
(iii) $R(A) \cap R(B) = \{0\}$, $N(A) \cap N(B) = \{0\}$,
(iv) $A - B,\ A^* + AB + B^*$ are nonsingular.

The following theorem provides necessary and sufficient conditions for the nonsingularity of a linear combination of powers of two commuting generalized projectors.

Theorem 5. [6] Let $A, B \in \mathbb{C}^{n\times n}$ be generalized projectors, and let $k, l \in \mathbb{N}$, $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ such that $c_1^3 + c_2^3 \neq 0$. Then $c_1 A^k + c_2 B^l$ is nonsingular if and only if $(I_n - P_{R(A)})B + P_{R(A)}$ is nonsingular.

The theorem below characterizes the situation where the invertibility of a linear combination of generalized projectors is equivalent to the invertibility of one projector, when their difference belongs to the class of commuting generalized projectors.

Theorem 6. [6] Let $A, B \in \mathbb{C}^{n\times n}$ be generalized projectors such that $B - A \in C_n^{GP}$, and let $k, l \in \mathbb{N}$, $c_1, c_2 \in \mathbb{C}$, $c_2 \neq 0$, $c_1^3 + c_2^3 \neq 0$. Then $c_1 A^k + c_2 B^l$ is nonsingular if and only if B is nonsingular.

Consequently, necessary and sufficient conditions for the invertibility of the sum of generalized projectors, whose difference is itself a generalized projector, are derived. The corresponding inverse of the sum is also characterized.

Corollary 7. [6] Let $A, B \in \mathbb{C}^{n\times n}$ be generalized projectors such that $B - A \in C_n^{GP}$. Then $A + B$ is nonsingular if and only if B is nonsingular and

$$(A+B)^{-1} = U\begin{bmatrix} \frac{1}{2}K^{-1} & 0 \\ 0 & G^{-1} \end{bmatrix}U^* = U\begin{bmatrix} \frac{1}{2}K^* & 0 \\ 0 & G^* \end{bmatrix}U^*, \qquad (4)$$

where $U \in \mathbb{C}^{n\times n}$ is unitary, $K \in \mathbb{C}^{r\times r}$ is such that $K^3 = I_r$, $K^* = K^{-1}$, and and $G \in \mathbb{C}^{(n-r)\times(n-r)}$ is an invertible generalized projector.

The form (4) can also be expressed in a more compact form. Namely, since

$$AA^\dagger = U\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}U^*,$$

and

$$I_n - AA^\dagger = U\begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix}U^*,$$

it follows that

$$U\begin{bmatrix} \frac{1}{2}K^{-1} & 0 \\ 0 & G^{-1} \end{bmatrix}U^* = U\begin{bmatrix} \frac{1}{2}K^* & 0 \\ 0 & G^* \end{bmatrix}U^* = \frac{1}{2}A^2 + (I_n - AA^\dagger)B.$$

Therefore,

$$(A+B)^{-1} = \frac{1}{2}A^2 + (I_n - AA^\dagger)B.$$

Example 2. Let us consider two generalized projectors $A, B \in \mathbb{C}^{3\times 3}$ defined by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \omega = e^{\frac{2\pi i}{3}}.$$

By direct calculation, we obtain $A^2 = A^* = A$, hence, $A \in C_n^{GP}$. Since,

$$B^2 = \begin{bmatrix} \omega^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B^* = \begin{bmatrix} \overline{\omega} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and since $\omega^2 = \overline{\omega}$, we also have $B^2 = B^*$, so $B \in C_n^{GP}$. The difference

$$B - A = \begin{bmatrix} \omega - 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

satisfies $(B-A)^2 = (B-A)^*$, thus $B - A \in C_n^{GP}$. All assumptions of Corollary 7 are therefore fulfilled. Using the relation

$$(A+B)^{-1} = \frac{1}{2}A^2 + (I_3 - AA^\dagger)B,$$

we compute $A^2 = A$, $A^\dagger = A$,

$$(I_3 - AA^\dagger) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence,

$$(A+B)^{-1} = \frac{1}{2}A^2 + (I_3 - AA^\dagger)B = \begin{bmatrix} \frac{1}{2}\omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

On the other hand, it is

$$A + B = \begin{bmatrix} 1+\omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $1 + \omega \neq 0$, the matrix $A + B$ is nonsingular. The inverse is easily found to be

$$(A+B)^{-1} = \begin{bmatrix} \frac{1}{2}\omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which agrees with the inverse computed above.

It should be noted that if the difference of two generalized projectors A, B is again a generalized projector, then their difference $B - A$ is singular. Indeed, in this case

$$B - A = U\begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix}U^*,$$

so it is evident that the difference is singular.

Another very useful representation of generalized inverses is the following: any generalized projector $A \in \mathbb{C}_r^{n\times n}$ has the form

$$A = U diag(\lambda_1, \lambda_2, \dots, \lambda_n)U^*,$$

where U is a unitary matrix and $\lambda_j \in \{0, 1, \omega, \varpi\}$, where $\omega = \mathrm{e}^{\frac{2\pi i}{3}}$, are the eigenvalues of A. Using this representation, one can derive necessary and sufficient conditions for the invertibility of linear combinations such as $c_1A + c_2B + c_3C$ with $BC = 0$, and $c_1I_n + c_2A + c_3B$ with $AB = 0$.

Theorem 8. [5] Let $c_1, c_{2,} c_{3,} \in \mathbb{C} \setminus \{0\}$ such that $c_1^3 + c_2^3 \neq 0$, and $c_1^3 + c_3^3 \neq 0$. If $A, B, C \in \mathbb{C}^{n\times n}$ are commuting generalized projectors such that $BC = 0$, then the following conditions are equivalent:
(i) $c_1A + c_2B + c_3C$ is nonsingular,
$(ii) B^3 + C^3 + A(I_n - B^3 - C^3)$ is nonsingular,
(iii) $rank\big(A(I_n - B^3 - C^3)\big) = n - \big(rank(B) + rank(C)\big)$.

Theorem 9. [5] Let $c_1, c_{2,} c_{3,} \in \mathbb{C}$, $c_1 \neq 0$, $c_1^3 + c_2^3 \neq 0$, and $c_1^3 + c_3^3 \neq 0$. If $A, B \in \mathbb{C}^{n\times n}$ are commuting generalized projectors such that $AB = 0$, then $c_1I_n + c_2A + c_3B$ is nonsingular and

$$\begin{aligned}(c_1I_n + c_2A + c_3B\,)^{-1} &= \frac{1}{c_1^3 + c_2^3}(c_1^2A^3 - c_1c_2A \\ &+ c_2^2A^2) \\ &+ \frac{1}{c_1^3 + c_3^3}(c_1^2B^3 - c_1c_3B \\ &+ c_3^2B^2) \\ &+ \frac{1}{c_1}(I_n - B^3 - C^3).\end{aligned}$$

CONCLUSION

In this paper, we have studied the invertibility for the linear combinations of generalized projectors and provided necessary and sufficient conditions for the invertibility of their linear combinations. Explicit forms of the inverses of such combinations were derived for the cases in which they are invertible. An important conclusion is that the invertibility of a linear combination of generalized projectors can be explicitly determined by the complex coefficients, in particular, when the sum of their cubes is nonzero. These results not only generalize known properties of classical projectors but also provide a systematic framework for analyzing linear combinations of generalized and hypergeneralized projectors in terms of their algebraic structure.

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