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A NONLINEAR CAPACITOR-BASED INTEGRATE-AND-FIRE NEURON MODEL

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Abstract

Neuron models have been extensively studied, and various formulations have been proposed in the literature to capture complex neuronal dynamics. While the Nobel Prize-winning Hodgkin-Huxley model offers high physiological accuracy, its mathematical complexity limits its computational efficiency in large-scale network simulations. Conversely, simplified paradigms such as the classical Leaky Integrate-and-Fire (LIF) model are widely preferred, yet they often fail to emulate critical features like spike-frequency adaptation without incorporating additional state variables or complex differential equations. In this study, we propose a novel non-linear Leaky Integrate-and-Fire neuron model driven by a voltage-dependent dynamic capacitance function C(v_C)=C_0+Kv_C (t). By embedding non-linearity directly into the capacitive node, the model inherently captures adaptive firing dynamics and frequency modulation under sustained current stimulation while preserving low computational overhead. Furthermore, exact closed-form analytical expressions for the triggering time and oscillation period are derived using the Lambert W function. Comprehensive numerical simulations demonstrate that the proposed model effectively replicates key adaptive characteristics, threshold-triggered resetting, and constant refractory period dynamics. These results demonstrate that the non-linear capacitive LIF model serves as a computationally efficient, analytically tractable, and versatile building block for neuromorphic hardware design and large-scale neural network simulations.

Keywords: *Nonlinear capacitor, voltage-dependent capacitor, neuron model, leaky integrate-and-fire, circuit analysis, circuit modeling.*

INTRODUCTION

Neuron modeling has been extensively researched, giving rise to various analytical frameworks over the years [1, 2]. Owing to its minimal computational overhead, the integrate-and-fire (IF) concept originally introduced by Louis Lapicque continues to be a cornerstone in solving diverse neuroscience problems [1, 2]. Lapicque's initial framework eventually paved the way for the biologically detailed Hodgkin-Huxley model [3], with various physical emulators developed in the literature [4]. Although the Hodgkin-Huxley formulation accurately replicates physiological mechanisms and complex neural activity, its high mathematical complexity makes it less ideal for large-scale network simulations. Consequently, simplified neuron abstractions are generally preferred when detailed individual ionic dynamics are not the primary focus. Among the numerous variations of IF models proposed in the literature, the leaky integrate-and-fire (LIF) model remains the most popular; it incorporates a leakage mechanism where the membrane potential gradually decays in the absence of external stimuli. Other non-linear alternatives, such as the FitzHugh-Nagumo [4, 5] and Izhikevich models [6], have gained traction due to their ability to exhibit richer computational dynamics. Nevertheless,

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standard LIF implementations remain significantly more straightforward and computationally efficient. A key limitation of the classical LIF neuron is its inability to demonstrate spike-rate adaptation (i.e., the progressive reduction in firing frequency during prolonged current injection). Conventional methods address this by adding secondary differential equations to dynamically adjust the firing threshold or introduce an adaptation current following each action potential [7]. Alternatively, incorporating non-linear circuit elements directly into the cellular membrane architecture can capture these adaptive responses without elevating the system's differential order [8]. Non-linear capacitive elements—whose capacitance changes non-linearly as a function of terminal voltage—have gained substantial traction in energy storage technologies, power electronics, and high-frequency circuit design [9–15]. A prominent real-world example is the electrochemical double-layer capacitor (supercapacitor), whose voltage-dependent capacitance can be accurately expressed as C (v_C)=C_0+Kv_C [9, 10], where C_0 denotes the baseline linear capacitance and K serves as the voltage-sensitivity parameter [17, 18]. Similar non-linear voltage-dependent capacitive phenomena are widely observed in semiconductor varactors, ferroelectric ceramic capacitors, and modern solid-state devices [12]. Furthermore, non-linear transient dynamics have been rigorously investigated across various domain applications, including diode circuits with parabolic characteristics [18], non-linear capacitance charge-discharge profiles [19], and memristive switching networks [20–22]. The neuron model presented in [23] is easily implementable and computationally straightforward to solve compared to traditional fractional-order models in the literature. Furthermore, this model effectively captures the adaptive firing response characteristic of biological neurons, aligning with results reported in other fractional-order neuron circuits [24–27].

Despite extensive research on voltage-dependent capacitors in energy storage and signal processing, their application within simplified spiking neuron models has not yet been explored. To the best of our knowledge, an LIF neuron model governed by a voltage-dependent non-linear capacitor of this form has not been previously investigated in the literature. The main contributions of this work are summarized as follows: We introduce a novel non-linear LIF neuron model driven by a voltage-dependent dynamic capacitance C(v_C)=C_0+Kv_C. Furthermore, we demonstrate that introducing non-linearity directly into the capacitive node allows the model to inherently exhibit adaptive firing dynamics and frequency modulation without increasing the differential order of the system. Finally, we derive exact closed-form analytical solutions for the triggering time and oscillation period utilizing the Lambert W function.

THE NON-LINEAR VOLTAGE-DEPENDENT CAPACITOR MODEL

The capacitance of the non-linear capacitor is voltage-dependent and can be expressed as:

$$C(v_C) = C_0 + K|v_C(t)| \quad (1)$$

where C_0 is the baseline linear capacitance, K is the voltage coefficient of the non-linear capacitance, and $v_C(t)$ represents the instantaneous capacitor voltage.

For positive operating voltages ($v_C(t) \geq 0$), the accumulated charge $q_C(t)$ and the dynamic differential capacitance $C(v_C)$ are simplified to:

$$q_C(t) = C_0 v_C(t) + K v_C^2(t) \quad (2)$$

and

$$C(v_C) = C_0 + K v_C(t) \quad (3)$$

Taking the time derivative of the charge expression yields the total capacitive current $i_C(t)$:

$$i_C(t) = \frac{dq_C(t)}{dt} = \frac{d}{dt}\left(C_0 v_C(t) + K v_C^2(t)\right) = \left(C_0 + 2K v_C(t)\right)\frac{dv_C(t)}{dt} \quad (4)$$

THE PROPOSED NON-LINEAR VOLTAGE-DEPENDENT CAPACITOR LIF NEURON MODEL

In this section, a novel non-linear voltage-dependent capacitor-based LIF neuron model is proposed. The equivalent electrical circuit schematic of the neuron model is illustrated in Figure 1. The circuit consists of a ideal switch (S), a constant current source (I_S), a parallel leakage resistor (R), and a non-linear dynamic capacitor (C). The current injected into the circuit is divided between the parallel leakage resistor R and the non-linear capacitor C, which prevents the capacitor voltage from rising instantaneously. The charging dynamics and transient response of the capacitor depend directly on the circuit parameters K, R, I_S, and C_0. The switch S is utilized to rapidly discharge C whenever the membrane potential reaches a specified threshold voltage (V_{th}), registering a spike event. Immediately following the reset, switch S opens to allow integration to resume. Consequently, the integrate-and-fire operation of the neuron continues periodically until the current injection is ceased.

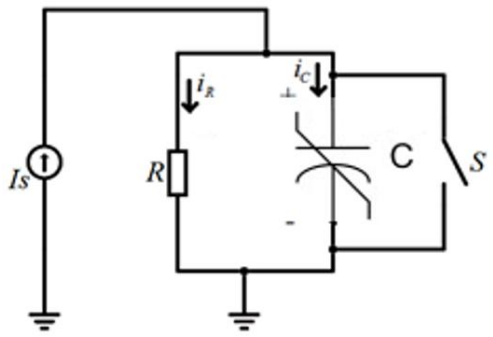


Figure 1. *The proposed nonlinear capacitor-based neuron model*

When an input current (I_S) flows through the neuron, the following differential equation can be written using Kirchhoff's laws:

$$I_S = i_C(t) + i_R(t) = (C_0 + 2K v_C(t))\frac{dv_C(t)}{dt} + \frac{v_C(t)}{R} \quad (5)$$

After multiplying each term with R and rearranging terms, derivative of the capacitor voltage ($v_C(t)$; called *membrane potential* hereafter) can be found as following:

$$I_S R = R(C_0 + 2K v(t))\frac{dv_C(t)}{dt} + v_C(t) \quad (6)$$

Assuming the CFD neuron's membrane potential is zero at t=0 s ($v_C(0) = 0$ V) and using the main branch of Lambert W function, W_0, the solution for $v_C(t)$ can also be found as

$$v_C(t) = RI_S - (C_0/2K + RI_S)W_0\left(-\frac{e^{\frac{\left(-\frac{t}{2RK} - (C_0/2K + RI_S)\ln(RI_S) - RI_S\right)}{(C_0/2K + RI_S)}}}{(C_0/2K + RI_S)}\right) \quad (7)$$

The membrane voltage has been expressed using Lambert W function, which is commonly found in science and engineering problems [28, 29]. At $t = t_{reset}$, the neuron voltage becomes equal to the threshold voltage. Therefore, the reset time (firing time) is found as:

$$t_{reset} = 2RK\left((C_0/2K + RI_S)\ln\left((-V_{TH} + RI_S)/(RI_S)\right) - V_{TH}\right) \quad (8)$$

There are 5 circuit parameters. One of them affects the shape of the spike. If the refractoriness is also assumed, a dead time t_{reset} after a spike can be used, which would result an inter-spike interval as following:

$$T = t_{ref} + t_{reset} = t_{ref} + 2RK\left((C_0/2K + RI_S)\ln\left((-V_{TH} + RI_S)/(RI_S)\right) - V_{TH}\right) \quad (9)$$

Then, the firing rate for current injection can be estimated as:

$$r = \frac{1}{T} = \frac{1}{t_{ref} + 2RK\left((C_0/2K + RI_S)\ln\left(\frac{(-V_{TH} + RI_S)}{RI_S} - V_{TH}\right)\right)} \quad (10)$$

With a continuous current injection, the instantaneous firing rate depends on the parameters, t_{ref}, R, K, C_0, V_{TH}, and I_S.

SIMULATION RESULTS

To evaluate the dynamic response of the proposed non-linear capacitive circuit, numerical simulations were performed using the parameters given in Table 1. The simulation parameters were set according to the scaled values derived from literature specs. Upon reaching the threshold voltage ($V_c(t) = V_{th}$), the capacitor voltage is instantaneously reset to 0 V and held at zero for a refractory period of $t_{ref} = 1\ \mu$s, during which charging is temporarily inhibited. Governing Equation and Reset Condition are given in Table 2. MATLAB codes have been written for this purpose.

Table 1. *The simulation parameters.*

Parameter	*Symbol*	*Value*	*Unit*
Initial Voltage	$V_c(0)$	*0*	*V*
Source Current	I_S	1×10^{-9}	*A*
Threshold Voltage	V_{th}	*2*	*V*
Refractory Period	t_{ref}	1×10^{-6}	*s*
Base Capacitance (C_0)	C_0	1×10^{-12}	*F*
Capacitance Coeff. (K)	*K*	1×10^{-13}	*F/V*
Resistance	*R*	10×10^{6}	*Ω*
Time Step	*dt*	1×10^{-9}	*s*
Simulation Time	t_{end}	20×10^{-6}	*s*

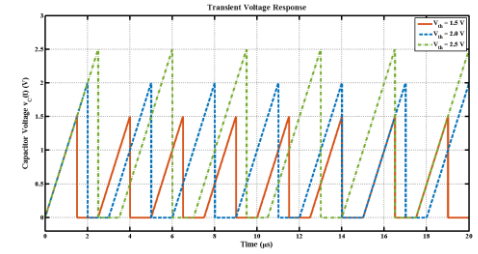
Table 2. *Governing Equation & Reset Condition*

Description	Expression
Differential Equation	$\frac{dv_C(t)}{dt} = \frac{I_S R - v_C(t)}{R(C_0 + 2Kv_C(t))}$
Reset Condition	$v_C(t) \geq 2$ V $\Rightarrow v_C(t) = 0$ V for t_{ref}

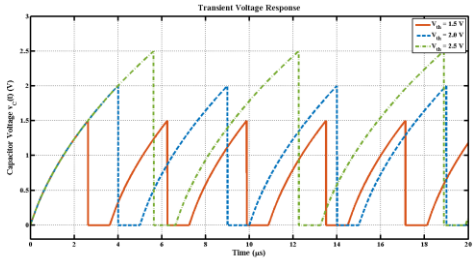
As illustrated in Figure 2.a, the transient voltage response of the dynamic capacitive node exhibits periodic relaxation oscillations under varying threshold voltage (V_{th}) parameters. For *K=0 F/V*, the non-linear capacitance turns into a linear time-invariant capacitance $C(v_C) = C_0 + Kv_C = C_0$. When a constant current I_S charges the capacitor the instantaneous voltage $v_C(t)$ increases monotonically aa a ramp until it reaches the specified threshold level V_{th}. Upon crossing V_{th}, the capacitor voltage is rapidly pulled down to 0 V within the same simulation step. Following the reset event, the circuit remains inactive at 0 V for a predefined refractory period ($t_{ref} = 1\ \mu$s), effectively modeling the refractory dead-time observed in biological neurons and leaky integrate-and-fire (LIF) oscillatory systems. A comparative evaluation across the three threshold values reveals two key dynamics. First, regarding frequency modulation, a lower threshold voltage ($V_{\text{th}} = 1.5$ V, red solid line) leads to shorter charging intervals, resulting in a higher spike/oscillation frequency. Conversely, increasing the threshold to $V_{\text{th}} = 2.5$ V (green dash-dotted line) extends the charging time, thereby decreasing the oscillation frequency. Second, in terms of refractory uniformity, across all cases ($V_{\text{th}} = 1.5$ V, 2.0 V, 2.5 V), the duration of the zero-voltage flat region remains constant at t_{ref}, confirming the independent operational stability of the refractory timer algorithm.

Figure 2.b demonstrates the transient capacitor voltage response $v_C(t)$ under three distinct threshold levels ($V_{\text{th}} = 1.5$ V, 2.0 V, and 2.5 V) for a non-zero capacitance coefficient $K = 5 \times 10^{-10}$ F/V. In contrast

to standard integrate-and-fire models where a constant capacitance ($K = 0$) yields a purely linear ramp charging trajectory, the introduction of a positive voltage dependence parameter ($K > 0$) causes the effective capacitance to increase with voltage. As a result, the instantaneous charging rate ($dv_C/dt = I_{\text{in}}/C(v_C)$) progressively decreases during the integration phase. This leads to a distinct sub-linear (concave) charging profile with a monotonically decreasing slope as the voltage approaches the threshold. Once $v_C(t)$ reaches the preset threshold voltage level V_{th}, an instantaneous reset mechanism is triggered, forcing $v_C(t)$ down to 0 V.



(a)



(b)

Figure 2: a) *Transient capacitor voltage dynamics $v_C(t)$ for K=0 F/V and three different threshold levels (V_{th} = 1.5 V, 2.0 V, and 2.5 V) showing threshold-triggered resetting and a constant refractory period (t_{ref} = 1 μs).*

b) *Transient capacitor voltage dynamics $v_C(t)$ for K=5.10^{-10} F/V and three different threshold levels (V_{th} = 1.5 V, 2.0 V, and 2.5 V) showing threshold-triggered resetting and a constant refractory period (t_{ref} = 1 μs).*

Following each firing event, the capacitor voltage remains clamped at zero for a fixed refractory period (T_{ref}) before initiating the next charging cycle. As observed from the curves, higher threshold voltages require a significantly longer time to reach V_{th}, not only because of the larger voltage span but also due to the severe attenuation of the charging rate at higher voltage levels. Consequently, increasing V_{th} markedly reduces the spiking/oscillation frequency while the constant refractory period remains identical across all cases.

Figure 3 illustrates the transient capacitor voltage dynamics ($v_C(t)$) over a 20 μs time window under a constant threshold voltage ($V_{\text{th}} = 2.0$ V) and a fixed refractory period for three distinct non-linear dynamic parameters ($K = 0$ F/V, $K = 5.0 \times 10^{-10}$ F/V, and $K = 1.0 \times 10^{-9}$ F/V). The simulation results reveal several key features of circuit dynamics. Across all configurations, the capacitor voltage $v_C(t)$ increases until it hits the prescribed threshold level ($V_{\text{th}} = 2.0$ V), whereupon it undergoes an instantaneous resetting back to 0 V. The parameter K strongly influences the charging slope: for $K = 0$ F/V (solid red curve), the capacitor exhibits the steepest slope, reaching V_{th} in approximately 2 μs. As K increases to 5.0×10^{-10} F/V (dashed blue curve) and 1.0×10^{-9} F/V (dash-dotted green curve), the charging rate noticeably decreases, extending the time required to reach the threshold to ~ 4 μs and ~ 6 μs, respectively. Following each reset event to 0 V, the system stays in an inactive state for a constant refractory duration (≈ 1 μs) before initiating the next charging cycle, demonstrating that the refractory delay remains invariant to changes in K. Consequently, higher values of K prolong the overall oscillation period, resulting in a lower spiking/resetting frequency.

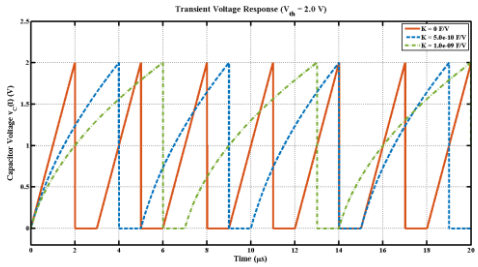


Figure 3: *Transient capacitor voltage dynamics $v_C(t)$ for V_{th} = 2.0 V and three different K values (K=0 F/V, K=5.10^{-10} F/V, and K=10^{-9} F/V) showing threshold-triggered resetting and a constant refractory period (t_{ref} = 1 μs).*

CONCLUSION

In this study, we demonstrated a non-linear capacitive neuron model driven by a voltage-dependent dynamic capacitance function $C(v_C) = C_0 + Kv_C(t)$. The proposed formulation offers a significantly lower computational complexity, making it easier and more efficient to implement in numerical solvers compared to conventional fractional-order and complex dynamic neuron models found in the literature. Furthermore, the model effectively captures the threshold-triggered resetting mechanism and constant refractory duration (t_{ref}) characteristics of physiological spike-frequency adaptation and bio-inspired oscillatory circuits. Nevertheless, the operational parameters—specifically the threshold voltage (V_{th}), the refractory duration (t_{ref}), and the non-linear capacitance coefficient (K)—must be carefully calibrated, as they jointly govern the charging dynamics, output firing rate, and overall system stability. Due to its low computational complexity and analytical clarity, this model also holds notable educational value for integration into academic curricula and simulation laboratories to teach threshold-based dynamics and non-linear circuit modeling.

Furthermore, as validated by our analytical derivations, the Lambert W function—defined as the multivalued inverse of the transcendental equation $we^w = z$, plays a vital role in providing exact closed-form solutions for the triggering time and oscillation period of the proposed dynamic circuit. While the Lambert W function is widely utilized in semiconductor thermal activation, diode modeling, and memristive systems, its application here provides a robust analytical framework for non-linear capacitive integrate-and-fire systems.

As part of future work, this non-linear model can be implemented into physical neuromorphic hardware circuits (such as CMOS or memristor-based integrated topologies), and the governing differential formulations can be extended to multi-neuron networks to explore collective synchronization, phase-locking, and complex spike-pattern encoding.

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